

Stochastic Sensitivity and Noise-Induced Bifurcations of Limit Cycles

Lev Ryashko

Ural Federal University, Ekaterinburg, Russia

Abstract

Deformations of limit cycles for dynamical systems forced by random disturbances are studied. A mathematical tool based on stochastic sensitivity analysis is shortly presented. A phenomenon of noise-induced bifurcation of supersensitive limit cycle of randomly forced Brusselator is discussed.

Keywords: Stochastic sensitivity, limit cycle, bifurcation

Introduction

Analysis of the forced nonlinear dynamical systems plays an important role for understanding of the various noise-induced phenomena in mechanics, electronics, biology and economics. Even small random disturbances may essentially change behavior of the dynamic system and generate noise-induced transitions, stochastic bifurcations, noise-induced order and chaos [1-7].

In the present paper, we study an influence of random disturbances on the auto-oscillations of nonlinear dynamical systems. Our analysis is based on stochastic sensitivity function technique and numerical simulations.

An essence of stochastic sensitivity functions technique is given in Section 1. In Section 2, a phenomenon of noise-induced bifurcation of supersensitive limit cycle of randomly forced Brusselator is presented and discussed.

1. Stochastic sensitivity function technique. Consider a nonlinear stochastic system

$$dx = f(x)dt + \varepsilon\sigma(x)dw, \quad (1)$$

where $w(t)$ is a standard Wiener process, ε is a noise intensity. It is supposed that the deterministic system (1) ($\varepsilon = 0$) has an exponentially stable limit cycle corresponding to T -periodic solution $x = \xi(t)$.

Under the random disturbances, trajectories of forced system leave the deterministic cycle and form a stationary probabilistic distribution around it.

Let l_t be a line orthogonal to the cycle at the point $\xi(t)$ ($0 \leq t < T$). In this case, for the Poincare section l_t , the stationary probabilistic distribution can be approximated as follows:

$$\rho_t(x, \varepsilon) \approx K \exp \left(-\frac{\|x - \xi(t)\|^2}{2m(t)\varepsilon^2} \right).$$

Here, $m(t) > 0$ is a T -periodic scalar stochastic sensitivity function satisfying the following boundary problem [8]

$$\dot{m} = a(t)m + b(t), \quad m(0) = m(T) \quad (2)$$

with T -periodic coefficients

$$a(t) = u^\top(t)(F^\top(t) + F(t))u(t), \quad b(t) = u^\top(t)S(t)u(t),$$

where $u(t)$ is a normalized vector orthogonal to $f(\xi(t))$,

$$F(t) = \frac{\partial f}{\partial x}(\xi(t)), \quad S(t) = \sigma(\xi(t))\sigma^\top(\xi(t)).$$

The explicit solution $m(t)$ of the problem (2) is given by

$$m(t) = g(t)(c + h(t)),$$

where

$$g(t) = \exp \left(\int_0^t a(s)ds \right), \quad h(t) = \int_0^t \frac{b(s)}{g(s)}ds, \quad c = \frac{g(T)h(T)}{1 - g(T)}.$$

The value

$$M = \max_{t \in [0, T]} m(t)$$

plays an important role in the analysis of stochastic dynamics near limit cycle. We shall consider M as a *stochastic sensitivity factor* of the cycle.

Stochastic sensitivity function technique was successfully used in control theory for the synthesis of regulators [9].

2. Stochastic Brusselator. Consider stochastically forced Brusselator

$$\begin{aligned} \dot{x} &= a - (b+1)x + x^2y + \varepsilon \dot{w} \\ \dot{y} &= bx - x^2y \end{aligned} \quad (3)$$

Here, ε is the intensity. For $b > \bar{b} = 1 + a^2$ the unforced Brusselator has a stable limit cycle. The value \bar{b} is a point of Andronov-Hopf bifurcation.

For the fixed $a = 0.2$, we have $\bar{b} = 1.04$. In Fig.1 the stochastic sensitivity factor is plotted for $b > \bar{b}$. On the considered interval the function $M(b)$ has a sharp high peak. As a result the function $M(b)$ has an essential overflow of values. By critical value of parameter b , one has $b_* = \arg \max_b M(b) = 1.064082$, $M(b_*) = 4.4 \cdot 10^{10}$. So, a cycle of Brusselator for $b = b_*$ is supersensitive even for small stochastic disturbances.

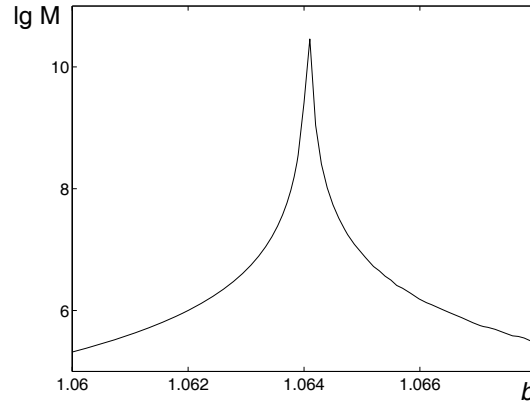


Fig. 1. Stochastic sensitivity factor for $a = 0.2$.

High sensitivity can imply unexpected qualitative changes in stochastic dynamics of Brusselator. In Figs.2,3 an influence of stochastic disturbances on supersensitive cycle with $b = 1.064082$ is demonstrated. In Fig.2 (left panel), an initial deterministic cycle of Brusselator is shown.

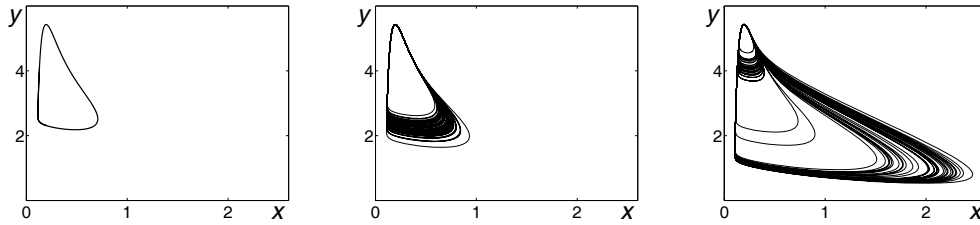


Fig. 2. Phase trajectories of Brusselator for $a = 0.2$, $b = 1.064082$: $\varepsilon = 0$ (left panel); $\varepsilon = 10^{-6}$ (middle panel); $\varepsilon = 10^{-4}$ (right panel).

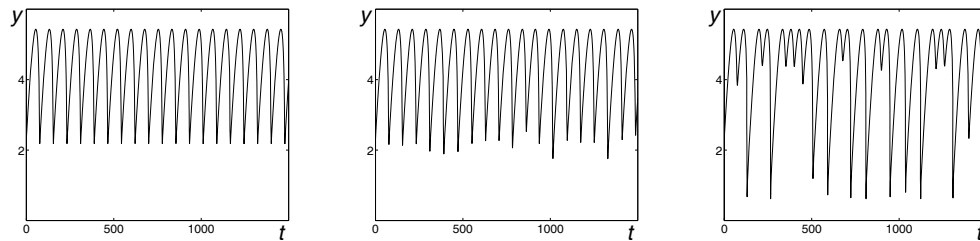


Fig. 3. Time series of Brusselator for $a = 0.2$, $b = 1.064082$: $\varepsilon = 0$ (left panel); $\varepsilon = 10^{-6}$ (middle panel); $\varepsilon = 10^{-4}$ (right panel).

Under the small background noise with $\varepsilon = 10^{-6}$, one can observe a considerable dispersion of random trajectories around unforced cycle on the bottom part (see Fig.2, middle panel). As noise intensity increases, qualitative change of this random distribution occurs. Indeed, for $\varepsilon = 10^{-4}$, random trajectories in the bottom part of stochastic cycle are repelled from the deterministic cycle. As a result, the bundle of random trajectories is splitted into two parts (see Fig.2, right panel).

Here, in Brusselator, the oscillations of small and large amplitudes coexist. This phenomenon is confirmed in detail by the corresponding time series in Fig.3.

Thus, using stochastic sensitivity function technique, we have found a supersensitive limit cycle of Brusselator. A high sensitivity level of this cycle causes a stochastic bifurcation of splitting.

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References

1. W. Horsthemke, R. Lefever, Noise-Induced Transitions, Springer, Berlin (1984).
2. J.B. Gao, S.K. Hwang, J.M.Liu, When can noise induce chaos? Phys. Rev. Lett., 82 (1998), 1132–1135.
3. P.S. Landa, P.V.E. McClintock, Changes in the dynamical behavior of nonlinear systems induced by noise, Phys. Rep., 323 (2000), 1–80.
4. V. S. Anishchenko, V. V. Astakhov, A. B. Neiman, T. E. Vadivasova, L. Schimansky-Geier, Nonlinear Dynamics of Chaotic and Stochastic Systems. Tutorial and Modern Development, Springer-Verlag, Berlin/Heidelberg (2007).
5. F. Gassmann, Noise-induced chaos-order transitions, Phys. Rev. E, 55 (1997), 2215–2221.
6. I. Bashkirtseva, L. Ryashko, P. Stikhin, Noise-induced backward bifurcations of stochastic 3D-cycles, Fluctuation and Noise Letters, 9 (2010), 89–106.
7. I. A. Bashkirtseva, L. B. Ryashko, H. Schurz, Analysis of noise-induced transitions for Hopf system with additive and multiplicative random disturbances, Chaos, Solitons and Fractals, 39 (2009), 72–82.
8. I. A. Bashkirtseva, L. B. Ryashko, Sensitivity analysis of the stochastically and periodically forced Brusselator, Physica A, 278 (2000), 126–139.
9. L. Ryashko, I. Bashkirtseva, On control of stochastic sensitivity, Automation and Remote Control, 69 (2008), 1171–1180.

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